II Superattracting case.

We study in details the extension of the Böttcher coordinate Φ on the basin of attraction associated to a superattracting germ.

Notice that to extend dynamically the map Φ $\left(\Phi \circ f = \tilde{f} \circ \Phi,\ \tilde{f}(z) = z^\delta \right)$

we should be able to give a meaning to: $\tilde{f}^{-n} \circ \Phi \circ f^n(z) = \left(\Phi \circ f^n(z) \right)^{\frac{1}{\delta^n}}$

This is not always possible (for example when $\exists\, z \in \mathcal{A}$, $f^n(z) = p$, or if $\mathcal{A}$ is not simply connected). ~~Idea: $\Phi = z(1 + \phi(z))$; $\Phi \circ f^n(z) = z$~~

But we can extend $|\Phi|$ on the whole basin, by setting:

$$|\Phi|(z) = \lim_{n \to +\infty} |\Phi(f^n(z))|^{\frac{1}{\delta^n}}.$$

We now study the locus where Φ is injective. To do so, we consider a local inverse Ψ defined on some small disk $\mathbb{D}_\varepsilon$.

Theorem: There exists a unique open disk $\mathbb{D}_r$ of maximal radius $0 < r \leq 1$ such that Ψ extends holomorphically to $\Psi : \mathbb{D}_r \to \mathcal{A}_0$

If $r = 1$, then $\Psi : \mathbb{D}_1 \to \mathcal{A}_0$ is a biholomorphism, and $\mathcal{A}_0 \cap \mathcal{C}_f = \{p\}$.

If $r < 1$, then there exist another critical point $z \in \mathcal{A}_0 \cap \mathcal{C}_f$, $z \in \partial \Psi(\mathbb{D}_r)$.

Proof: As in the attracting case we can extend Ψ (by checking for example the radius of convergence of the power series Ψ at 0).

It ~~extent~~ can hence extend to $\mathbb{D}_r$ for some maximal radius $r \leq 1$.

(~~as~~ $r \leq 1$ since $w^{d^n} \not\to 0$ when $|w| = 1$)

7.8

- Ψ has no critical points in $\mathbb{D}_r$.

In fact, if $\exists w$, $\Psi'(w)=0$, then:

$$\Psi(w^\delta)=f(\Psi(w)) \Rightarrow \Psi'(w^\delta)=f'(\Psi(w))\cdot\Psi'(w)=0.$$

This would give a sequence of critical points accumulating to 0, which is impossible.

Hence Ψ is locally one-to-one (i.e. a covering map $\Psi:\mathbb{D}_r\to\Psi(\mathbb{D}_r)\subset A_0$), and $\{(w_1,w_2)\mid w_1\neq w_2,\ \Psi(w_1)=\Psi(w_2)\}\subset\mathbb{D}_r\times\mathbb{D}_r$ is a closed set.

We show that Ψ is actually 1-to-1.

The map $|\Phi|: A_0\to\mathbb{C}$ satisfies $|\Phi\circ\Psi(w)|=|w|$ on a small neighborhood of 0, and hence on $\mathbb{D}_r$ by analytic continuation.

Suppose $\Psi(w_1)=\Psi(w_2)$ with $w_1\neq w_2$.* Applying $|\Phi|$, we get $|w_1|=|w_2|$ *.

By taking w' close to w_1, we can find w'' close to w_2 so that $\Psi(w')=\Psi(w'')$ (because Ψ is a covering map)

by taking w', $|w'|<|w_1|$, we get a contradiction.

$\boxed{\text{* Take such } (w_1,w_2) \text{ so that } |w_j| \text{ is minimal (ok because of closedness)}}$

We have two cases now:

- $r=1$: then $A_0=\Psi(\mathbb{D})$: Suppose otherwise, then $\Psi(\mathbb{D})$ has some boundary point $z_0\in A_0$. Take $w_n\in\mathbb{D}$, $\Psi(w_n)\to z_0$, we deduce that $|\Phi(\Psi(w_n))|=|w_n|\to 1$.

$\Rightarrow|\Phi(z_0)|=1$, a contradiction because $\Phi(A_0)\subset\mathbb{D}$.

- $r<1$. In this case, the proof of existence of a critical point is analogous to the attracting case.

$\square$

Applications to polynomial dynamics

Let $P \in \mathbb{C}[z]$ be a polynomial, $\deg P = d \geq 2$.

Write $P(z) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_0$. Up to linear change of coordinates, we

may assume $a_d = 1$. (monic). (and $a_{d-1} = 0$, "centred" polynomial) ↗affine

Then ∞ is a superattracting fixed point and $J = \hat{\mathbb{C}} \setminus A_\infty$ ↖basin of attraction

The Böttcher coordinate of ∞ can be used to study the properties of $J(f)$ for $f \in \mathbb{C}[z]$

Def: The filled Julia set of $f \in \mathbb{C}[z]$ is the set $K(f)$

of points with bounded orbit: $K(f) = \hat{\mathbb{C}} \setminus A_\infty$.

Lemma: $\forall f \in \mathbb{C}[z]$, $\deg f \geq 2$, $K = K(f)$ is compact, with connected complement, and $\partial K = J(f)$.

$\mathring{K}$ is the union of all bounded components U of the Fatou set $\mathbb{C} \setminus J$.

(called (bounded) Fatou components). Any such component is necessarily

simply connected.

Proof: As noticed above, $K = \hat{\mathbb{C}} \setminus A_\infty$, and by previous results $\partial A_\infty = J(f)$.

Hence K is compact, and $\partial K = \partial A_\infty = J(f)$.

We must show that A_∞ is connected. Let U be any bounded Fatou component.

Let $r_0 \gg 0$ be such that $|f(z)| > \lambda |z|^d$, $\lambda > 1$, $|z| > r_0$. $\Rightarrow f^n(z) \to \infty$ $\forall z, |z| > r_0$.

Then $|f^n(z)| \leq r_0$ $\forall z \in U$, $\forall n \geq 0$.

otherwise, By the max. modulus principle, $\exists \tilde{z} \in \partial U \subset J$, $n \in \mathbb{N}: |f^n(\tilde{z})| > r_0$

$\Rightarrow f^n(\tilde{z}) \to \infty$, which is a contradiction ($\tilde{z} \in J(f)$).

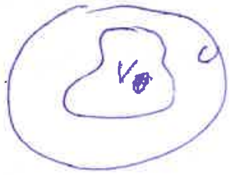
Thus every bounded Fatou component is in $K(f)$, and the unique unbounded

component is $\mathbb{C} \setminus K = \mathbb{C} \cap A_\infty$.

17.10

It remains to prove that U is simply connected.

Let γ be a ~~simple~~ closed curve lying in U, and V the bounded component of $\mathbb{C}\setminus\gamma$.



By the maximum modulus principle, $V \subset K(P)$.

In particular, $V \cap J(K) = \emptyset$, and since $\partial U \subset J(P)$, we get $V \subset U$, and U is simply connected. $\square$

Theorem (Connectivity of $J(P)$, $K(P)$ for polynomials)

Let $P \in \mathbb{C}[z]$, $\deg P = d \geq 2$.

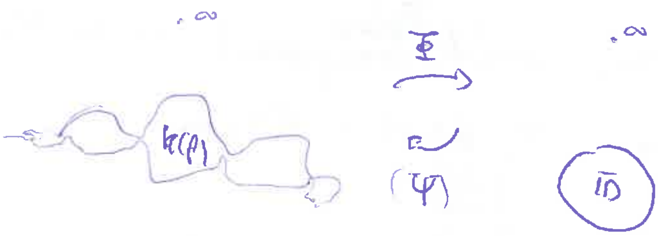
- If $(\mathcal{C}(P) \cap \mathbb{C}) \subset K$ (all critical points in $\mathbb{C}$ have bounded orbit), then $K(P)$ and $J(P)$ are connected, and A_∞ is conformally isomorphic to $\mathbb{D}$ under the Böttcher coordinate at ∞ (appropriately modified). $\hat{\Phi} : A_\infty \to \hat{\mathbb{C}}\setminus\overline{\mathbb{D}}$. $\hat{\Phi}\circ f = P_d \circ \hat{\Phi}$, $P_d(z) = z^d$.

- If at least one critical point of P belongs to $\mathbb{C}\setminus K(P)$ (has unbounded orbit $\Leftrightarrow \in A_\infty\setminus\{\infty\}$), then both $K(P)$ and $J(P)$ have uncountably many connected components.

Proof.

$P : \hat{\mathbb{C}} \circlearrowleft$ has a superattracting point at ∞ (of degree $d = \deg P$).

Consider the Böttcher coordinate at ∞, appropriately redefined so that Φ sends a small neighborhood of $\infty \in \hat{\mathbb{C}}\setminus K(P) = A_\infty(P)$ to a small neighborhood of ∞ in $\hat{\mathbb{C}}\setminus\overline{\mathbb{D}}$.



It satisfies:

$$\Phi \circ P(z) = (\Phi(z))^d.$$

7.11

1) Assume $\mathcal{A}_\infty \cap \mathcal{C}_f \cap \mathbb{C} = \emptyset$.

Then we have seen that the local inverse Ψ of Φ extends to a conformal isomorphism $\hat{\mathbb{C}} \setminus K \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Consider an annulus $A_{1+\varepsilon} = \{ z \in \mathbb{C} \mid 1 < |z| < 1+\varepsilon \}$.

Then $\Psi(A_{1+\varepsilon})$ is a connected set in $\mathbb{C} \setminus K(f)$, and $\overline{\Psi(A_{1+\varepsilon})} \supset J(f)$.

(being $\mathcal{A}_\infty = \mathcal{A}_\infty^\circ$ connected by the previous lemma (consequence of max. mod principle))

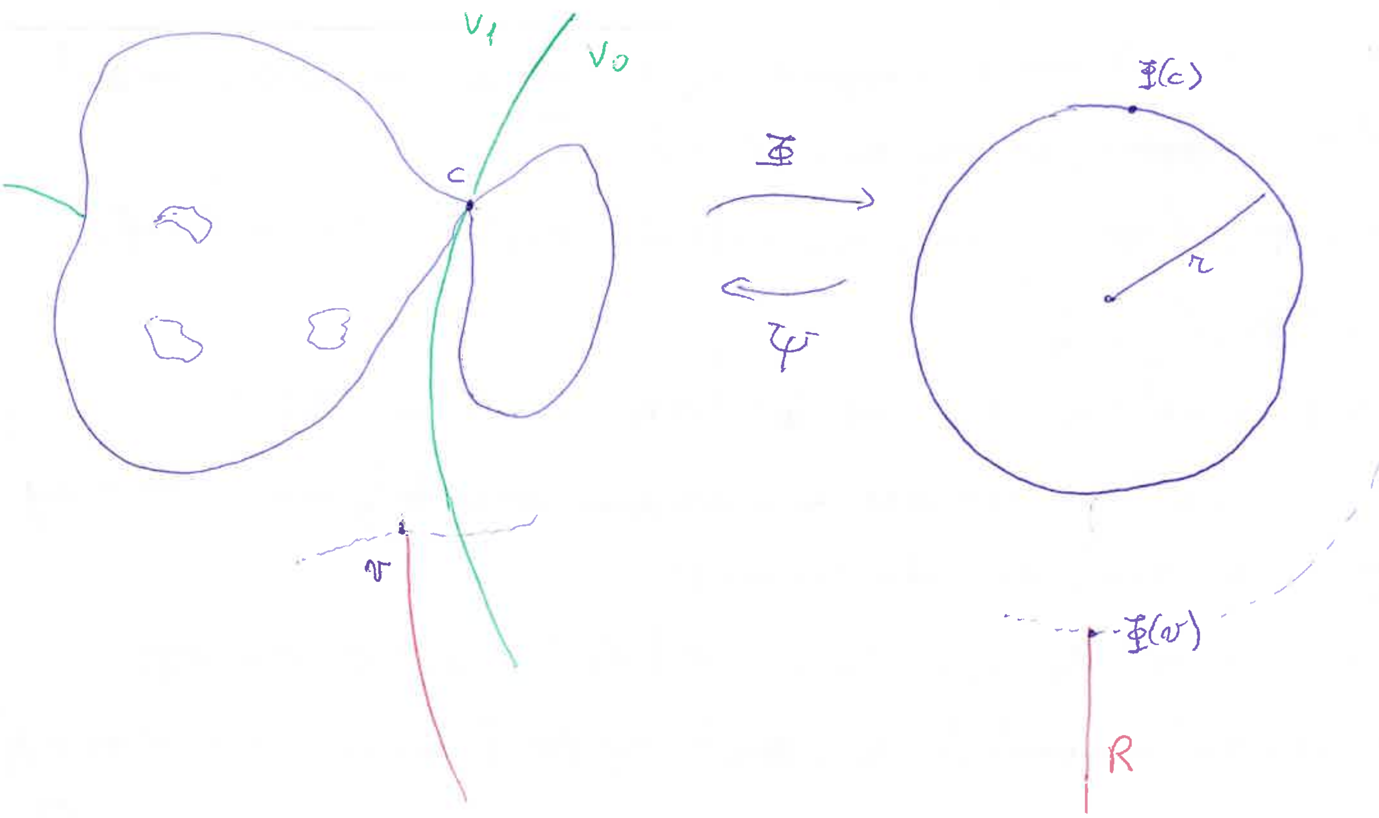
Then $J = \bigcap_{\varepsilon > 0} \overline{\Psi(A_{1+\varepsilon})}$ is also connected, and K also is (being $J = \partial K$ connected, K closed)

2) Suppose there is at least one critical point in $\mathbb{C} \setminus K(f) = \mathcal{A}(f) \setminus \{\infty\}$.

In this case, $\exists$ minimal $r > 1$ such that Ψ extends to a conformal isomorphism

$$\Psi : \mathbb{C} \setminus \overline{\mathbb{D}_r} \xrightarrow{\simeq} U \subset \mathbb{C} \setminus K(f). \qquad U = \Psi(\mathbb{C} \setminus \overline{\mathbb{D}_r})$$

Moreover, $\partial U \subset \mathbb{C} \setminus K$ is compact and contains at least one critical point of f.



Take c a critical point in ∂U. Set $v = f(c)$ its critical value, that belongs to U ($|\Phi(v)| = r^d > r$)

Let $R = [1, +\infty)\cdot\Phi(v)$ and $R' = \Psi(R)$.

R' is called "external ray" of v.

> Def: external ray: a set of the form $\Psi(R_\theta)$, $R_\theta = \{t\, e^{2\pi i \theta},\ \theta \in \mathbb{R},\ t \geq t_0\}$ ← closed
> fixed (θ) ($t > t_0$) ← open
> $t_0 \geq 1$.

Consider the full inverse image $f^{-1}(R') \subset \overline{U}$.

It consists of d distinct external rays, corresponding to the d distinct components of $\sqrt[d]{R} \subset \mathbb{C} \setminus \mathbb{D}_r$.

Each such external ray ends at some solution z of $f(z) = v$.

Being c a critical point and $f(c) = v$, at least two external rays, say R_1' and R_2', land at c, and $\mathbb{C} \setminus (R_1' \cup R_2')$ is the disjoint union of two connected open sets V_0, V_1.

Claim: $f(V_k)$ contains $\mathbb{C} \setminus R'$ $\forall k = 0, 1$.

In fact: $f(V_k)$ is a open set. We show that its boundary is contained in R'.

Let $\hat{w} \in \partial f(V_k)$. Pick $z_j \in V_k$ so that $f(z_j) \to \hat{w}$. $\{z_j\}$ is bounded (~~contains~~ (or we would have a subsequence $z_{j_h} \to \infty \Rightarrow \hat{w} = \infty$ contradiction).

$\Rightarrow$ up to taking a subsequence, we may assume $z_j \to \hat{z} \in \overline{V_k}$

If $\hat{z} \in V_k$, being f open, we would have $f(\hat{z}) = \hat{w} \in f(V_k)$. $\Rightarrow \hat{z} \in \partial V_k = R_1' \cup R_2'$

and $\hat{w} \in f(R_1' \cup R_2') \subseteq R'$.

Hence $f(V_k) \supseteq \mathbb{C} \setminus R' \supset K \supseteq J$. Set $J_k = J \cap V_k$. The $f(J_0) = f(J_1) = J$.

In particular $J = J_0 \sqcup J_1$ is disconnected, hence contains uncountably many connected comp.

~~Similarly~~ Moreover, $K = K_0 \sqcup K_1$ is also disconnected

Inductively, we can set $K_{\alpha_0 \ldots \alpha_n} = K_{\alpha_0 \ldots \alpha_{n-1}} \cap f^{-1}(K_{\alpha_n})$, and get uncountably many connected components for K. ((the same for J reproves a theorem we saw in §4)

□

Other applications to polynomial dynamics.

• The Green Function.

Let $f \in \mathbb{C}[z]$ be a polynomial of degree $d \geq 2$, and Φ the Böttcher coordinate of infinity as defined above, Φ: defined from a nbhd of $\infty \in \hat{\mathbb{C}} \setminus K$ to a nbhd of $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

The function $|\Phi|$ extends continuously to $A_\infty = \hat{\mathbb{C}} \setminus K$, taking values $|\Phi(z)| > 1$ (consider this map on $\mathbb{C} \setminus K \to \mathbb{C} \setminus \overline{\mathbb{D}}$).

Definition: The Green Function (or the canonical potential function) associated to f (or better, $K(f)$) is the map $G : \mathbb{C} \to [0, +\infty)$ defined by

$$G(z) = \begin{cases} 0 & z \in K \\ \log|\Phi(z)| = \lim_{k \to \infty} \frac{1}{d^k} \log|f^{\circ k}(z)| > 0 & , z \in \mathbb{C} \setminus K. \end{cases}$$

for $|z| >> 1$

($\log|\Phi_0|$ is harmonic, and use (*))

Notice that G is continuous everywhere, and harmonic on $\mathbb{C} \setminus J(f)$.

Moreover, it satisfies: $G(f(z)) \overset{(*)}{=} d \cdot G(z)$ $\quad (G \circ f = dG)$.

The curves $G = \text{constant} > 0$ are called equipotentials.

• Mandelbrot set.

Consider polynomials of degree 2. Up to affine conjugacy, they are all given by $P_c(z) = z^2 + c$, $c \in \mathbb{C}$. Notice that $\mathcal{C}(P_c) = \{0\}$.

The set $\mathcal{M} = \{c \in \mathbb{C} \mid (P_c^n(0))_n \text{ is bounded}\}$ is exactly the set of parameters c so that $J(P_c)$ is connected.

Local connectivity and external rays.

· Assume we are in the case of $P \in \mathbb{C}[z]$, $\deg P = d \geq 2$, and $J(P)$ connected.

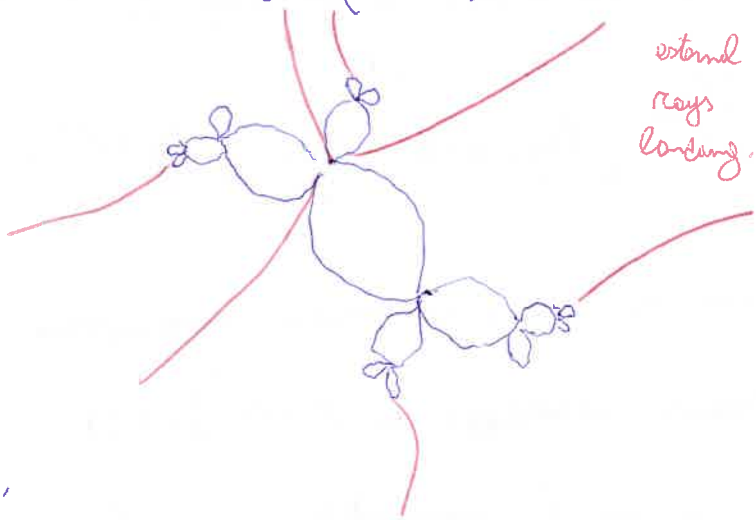
The Böttcher coordinate gives an isomorphism $\Phi : \mathbb{C} \setminus K(P) \to \mathbb{C} \setminus \overline{\mathbb{D}}$, of inverse ψ.

$\forall t \in \mathbb{R}/\mathbb{Z}$

Denote by $R_t = \{ \rho e^{2\pi i t} \mid \rho > 1 \}$. Its image $\psi(R_t) = R_t'$ is called the external ray associated to t.

Notice that, if $\tilde{P}(z) = z^d$, then $\tilde{P}(R_t) = R_{dt}$, and the action of $\tilde{P}$ on $\{R_t\} \simeq \mathbb{R}/\mathbb{Z}$, and hence of P on $\{R_t'\}$, is conjugated to
$$\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, \quad t \mapsto dt \pmod{\mathbb{Z}}.$$

Def: we say that the ray R_t' lands at a point $\gamma(t)$ $(\in J(P))$ if
$$\gamma(t) = \lim_{\rho \downarrow 1^+} \psi(\rho e^{2\pi i t})$$



Properties:

· If R_t lands at $\gamma(t)$, then R_{dt} lands at $\gamma(dt) = P(\gamma(t))$. Moreover, every $R_{\frac{t+j}{d}}$ lands to a preimage of $\gamma(t)$, and all such preimages are landing points.

· The set $\{ t \in \mathbb{R}/\mathbb{Z} \mid R_t \text{ does not land} \}$ has measure 0.

(Carathéodory)

Theorem: The following condition are equivalent.

· $\forall t \in \mathbb{R}/\mathbb{Z}$, R_t lands to a point $\gamma(t)$, and γ is C^0.

· $J(P)$ is locally connected ($\forall z \in J(P)$, $B(z,\varepsilon) \cap J$ is connected for $\varepsilon << 1$) ($\exists \varepsilon_n \downarrow 0$ s.t.)

· $K(P)$ is locally connected.

· $\psi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K$ extends continuously over $\partial \mathbb{D}$, sending $e^{2\pi i t} \mapsto \gamma(t) \in J(P)$.

In this case $\gamma : \mathbb{R}/\mathbb{Z} \to J(P)$ is semiconjugated to $t \mapsto dt$ in $\mathbb{R}/\mathbb{Z}$: $\gamma(dt) = P(\gamma(t))$.

See [Milnor, § 17,18] for further details: uses a topological theory of prime ends by Carathéodory.